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BOUNDARY ELEMENT METHOD AND MARKOV CHAIN MONTE CARLO FOR OBJECT LOCATION IN ELECTRICAL IMPEDANCE TOMOGRPHY

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Abstract − A Bayesian approach to object location in electrical tomography is presented. The direct problem, which is traditionally modelled by domain discretization methods such as finite-element and finite-difference methods, is reformulated using a straightforward but ultimately powerful implementation of the boundary-element method.

1. INTRODUCTION

All tomography techniques aim to reconstruct the interior of an object using measurements taken outside or on the boundary of the object. Such techniques are widely used in geophysical, industrial and medical investigations. In electrical tomography, voltages are recorded between electrodes attached to the boundary and the objective is to reconstruct the interior electrical conductivity distribution. Most commonly used approaches to reconstruction are based on domain discretization, leading to ill-posed inverse problems. To obtain a stable and reliable solution requires regularization, which can be usefully viewed as inclusion of prior information. If the conductivity distribution is given, then voltages on the boundary can be found using Maxwell's equations, and appropriate boundary conditions. This is the direct problem or forward solution. In practice this is done numerically, usually using the finite-element method (FEM). Despite the generality of FEM, and its widespread use, it does have substantial unappealing features for certain types of problem. In particular, it can be computationally inefficient, and good domain discretization is often problematic with very fine meshes and adaptive mesh refinement needed to maintain accuracy where conductivity changes rapidly. In contrast, the boundary-element method (BEM) requires only boundary discretization and can provide the solution and its derivative at any point in the domain. Furthermore, the Bayesian Markov chain Monte Carlo (MCMC) approach requires only multiple forward solutions as opposed to an explicit inverse formation. This paper investigates the use of a novel BEM/MCMC approach for simulated EIT data. For examples of FEM-based approaches see [8, 7].

2. THE BOUNDARY ELEMENT METHOD

Consider the cross section of a pipe with smooth exterior bounday $\partial\Omega$ and interior domain Ω. In the direct problem, the conductivity, $\sigma(s), s \in \Omega$, in the domain is specified and the electric field potential, $\phi(t), t \in \partial\Omega$, is calculated at points on the domain boundary. The potentials are then used to find voltages, that is potential differences. Within the domain the conductivity and potential are related by Maxwell's equation

$$\nabla \cdot (\sigma(s)\nabla\phi(s)) = 0. \tag{1}$$

For domains with a constant conductivity distribution, $\sigma(s) = \sigma$ for all s, Maxwell's equation simplifies to Laplace's equation,

$$\nabla^2\phi = 0. \tag{2}$$

This problem can be solved using the BEM to give the potential and its derivative at any point on the boundary or in the domain. The BEM is an interesting alternative to the more commonly used numerical approaches such as finite element methods. See [2] and [3] for details.

For any realistic problem exact solution of the boundary problem is not possible, but numerical solution is feasible. Domain discretization approaches are extremely popular however, the computational time required for their implementation is often large, particularly when fine meshes are employed. Since the proposed statistical method for the inverse problem requires a large number of solutions of the direct problem to be computed, it is necessary that a more expedient forward solution method be sought. The BEM provides an appealing, and often faster, alternative to domain discretization.

Briefly, the boundary-element approach is based upon the numerical solution of an integral equation which arises from having performed one integration of the Laplace equation via Green's theorem. The resulting *boundary integral equation* is,

$$\eta(s)\phi(s) = \int_{\partial\Omega} \{\phi_{\mathbf{n}}(s)G(s,s') + \phi(s)G_{\mathbf{n}}(s,s')\}d\Gamma, \tag{3}$$

where $G(s,s')$ is the free-space Green's function of the Laplace equation (also called a fundamental solution), $\phi(s)$ is the electric potential (voltage), and the subscript $\mathbf{n}$ denotes the derivative with respect to the outward pointing normal to the boundary. The 'location' function $\eta(s)$ takes the values

$$\eta(s) = \begin{cases} \frac{1}{2} & \text{if } s \in \partial\Omega \\ 1 & \text{if } s \in \Omega\backslash\partial\Omega \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

The key then is to discretize the domain boundary into boundary elements. Here constant elements are assumed, meaning that the boundary is approximated by piecewise linear segments, and that the potential on the boundary is piecewise constant. Other possibilities include linear and quadratic elements. The boundary $\partial\Omega$, is discretised into N boundary elements numbered counter-clockwise. The vectors of potentials and the normal derivatives on the elements are denoted $\{\phi\}$ and $\{\phi'\}$. Each element is related to the others by two influence integrals which are then assembled into $N \times N$ influence matrices H and G which, along with $\{\phi\}$ and $\{\phi'\}$, are related by the linear system, $[H]\{\phi\} = [G]\{\phi'\}$, see [3].

The BEM has the advantage, over the FEM, that one of the two required integrations of the governing partial differential equation is completed analytically, by the application of Green's theorem, reducing the number of unknowns from $O(N^2)$ to $O(N)$. The resulting boundary integral equation is then solved, for the remaining unknowns, via a numerical approach.

This paper considers problems where the electrical conductivity is piecewise constant and hence the domain must be subdivided into multiple homogeneous subdomains. This leads to a *multi-zone* or *composite body* problem. Within each of the subdomains, Maxwell's equation simplify to Laplace's equation. Hence the multi-zone problem can then be solved by applying the BEM to each of the subdomains separately.

In addition to the exterior boundary conditions there is also continuity of potential and continuity of flux across the interior interfaces. Therefore, in the case of two subdomains, e.g. a pipe (Ω_1) and an anomaly inside (Ω_2),

$$\nabla^2\phi_1 = 0 \text{ in } \Omega_1 \tag{5a}$$

$$\nabla^2\phi_2 = 0 \text{ in } \Omega_2. \tag{5b}$$

The boundary of Ω_1 will comprise electrodes E_k $(k = 1, 2, \cdots, K)$,whilst the remainder of the external boundary is insulated. Following, for example [6], the boundary conditions on $\partial\Omega_1$ are

$$\int_{E_k} \sigma_1 \frac{\partial\phi_1}{\partial\mathbf{n}} ds = I_k, \tag{6}$$

$$\sigma_1 \frac{\partial\phi_1}{\partial\mathbf{n}}\Big|_{\partial\Omega_1\backslash\cup_{k=1}^{K}E_k} = 0, \tag{7}$$

where I_k is the amplitude of the current injected through electrode E_k and as before $\mathbf{n}$ is the outward unit normal to the boundary. For ease of exposition, the contact impedances are neglected. On the boundary between Ω_1 and Ω_2, the continuity of the potential and the flux are enforced. Therefore there are sufficient boundary conditions for the direct problem to be solved.

Combining the separate matrix systems for the subdomains, then re-arranging the unknowns and knowns into vectors X and Y respectively, and appropriate rearrangement (of which there is a good example in [3]) of H and G into, A and B leads to $AX = BY$, and hence

$$\hat{X} = (A^T A)^{-1} A^T BY, \tag{8}$$

estimates the unknowns, provided the problem is well-conditioned. The required components of X corresponding to the boundary voltages, can then be identified.

3. BAYESIAN MODELLING

For a given conductivity distribution, σ, it is possible to calculate the corresponding electrode voltages $V^*(\sigma)$ using the above BEM, however, due to measurement errors the observed voltages will be noisy versions of the calculated values leading to measured voltages $V = \{V_1, \ldots, V_n\}$. Assuming independent Gaussian errors leads to the likelihood

$$p(V|\sigma) = \frac{1}{(2\pi\tau^2)^{(n/2)}} \exp\left\{-\frac{1}{2\tau^2}||V - V^*(\sigma)||^2\right\}$$

where τ^2 is the noise variance. Here voltages $V^*(\sigma)$ are calculated based on the given conductivity distribution σ. The parameters to be estimated, however, are the coordinates of the interface, and the conductivity parameters for the separate subdomains.

In the implementation a single inclusion is considered, with conductivity σ_1, and background conductivity σ_0. The centres of the boundary elements of the inclusion are described in a *star-shaped* manner with overall center (μ_r, μ_θ), in polar coordinates, and radii, $r = \{r_1, \ldots, r_m\}$, at m equal angles. The conductivity distribution is then a function of these, and hence V^*.

If it is assumed that no further information is available then estimation of the model parameters would be based on the likelihood only, however, here additional information is included. The interior boundary is expected to be smooth, and so a Gaussian distribution on second-order differences with small variance, ν^2, is used

$$p(r) \propto \frac{1}{(2\pi\nu^2)^{(m/2)}} \exp\left\{-\frac{1}{2\nu^2}||r - \bar{r}||^2\right\}$$

where $\bar{r}$ is a vector of mean neighbouring radii. Notice that, since any constant change to the radii has no effect on the prior density, this is an improper prior, that is it has an infinite integral. However, the data will contain information about the overall size of the inclusion and so the posterior distribution will be proper, and estimation feasible.

The posterior distribution of any parameter set given the data is then the result of combining the prior and likelihood distributions by Bayes' theorem. Inference about the model parameters is then based on this posterior distribution. A Metropolis-Hastings algorithm is used to produce approximate samples from the posterior distribution, [5]. The Metropolis–Hastings algorithm simulates a Markov Chain for each of the parameters which are being estimated, in this case the location, shape and electrical conductivity of an inclusion. For details of Bayesian modelling and MCMC methods see, for example, [1] and [4].

4. SIMULATION RESULTS

Consider the conductivity distribution shown in Figure 1. This might represent the cross-section of a pipe containing water, with constant conductivity 1 Ωm, and a solid anomaly, with constant conductivity 5 Ωm. Although the exterior boundary discretisation is determined by the relative sizes of the electrodes and gaps, the same is not true for the interior interface. Here, 32 elements were used to produce a realistically smooth interface. To generate data, it is assumed that there are 8 electrodes, leading to the 49 voltages shown in Figure 2.

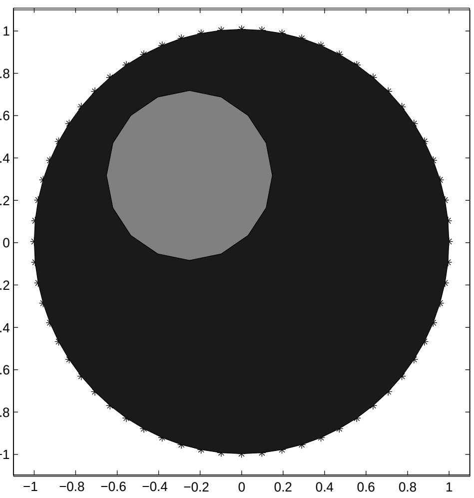

Figure 1: True conductivity, with boundary element discretisation.

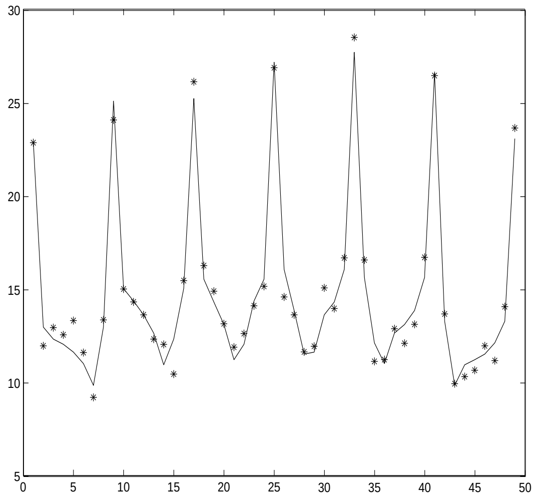

Figure 2: Simulated data (points) and noise-free values (line).

The MCMC algorithm was run from an arbitrary start state for 20,000 iterations, with the first 10,000 discarded, and then every 20th collected to produce a posterior sample of size 500. Parameter traces are shown in Figure 3; posterior summaries of mean and standard deviation in Table 1; parameter correlations are in Table 2; and the posterior conductivity distribution in Figure 4.

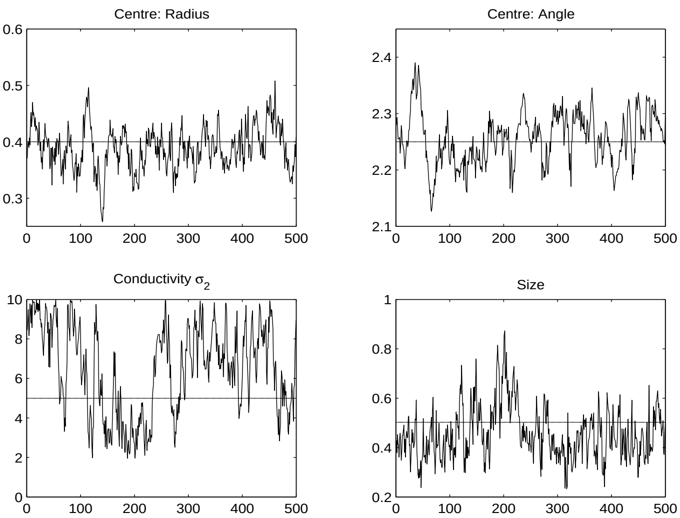

Figure 3: MCMC traces for the model parameters.

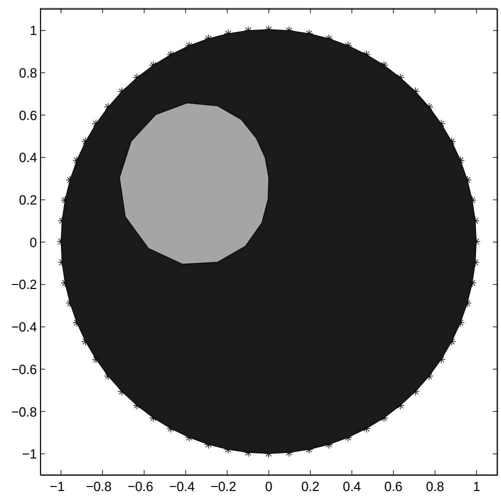

Figure 4: Conductivity distribution given by posterior mean parameters.

The traces indicate moderate autocorrelation within the chain despite the 1 in 20 sub-sampling. The center parameters fluctuate around the true values, producing estimates very close to the true values. Figure 4 shows the conductivity distribution from the posterior mean parameters, which is an excellent match to the true distribution in terms of location, shape and conductivity.

	True	Posterior mean	Posterior std dev
Centre, radius	0.40	0.389	0.123
Centre, angle	2.25	2.256	0.046
Conductivity	5.0	6.470	2.251
Size	0.50	0.456	0.107

Table 1: Posterior summary of model parameters.

Notice, from Table 1, that the angular component of the centre of the inclusion is more precisely estimated than the radial component. In contrast, the conductivity and size are less well recovered. From the traces it appears that when the conductivity is high the size is low. The correlations in Table 2 reinforce this with a substantial negative value, whereas other correlations are small.

	Centre radius	Centre angle	Conductivty
Centre, angle	0.13		
Conductivty	0.15	0.20	
Size	-0.25	-0.20	-0.54

Table 2: Estimated correlations between parameters.

5. CONCLUSIONS

These results indicate that the proposed boundary element Markov chain Monte Carlo (BEMCMC) approach is an exciting combination of the BEM for fast forward solution, and stochastic geometric modelling. The Bayesian approach has the flexibility to allow the incorporation of varied prior knowledge, and the MCMC algorithm allows flexible posterior distribution investigation and summary. Incorporation of process control and feedback modeling forms an interesting and industrially relevant extension to the work presented.

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